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SINGULAR COMPACTNESS AND THE NEVEU DECOMPOSITION

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ABSTRACT

The aim of this note is to determine the Neveu decomposition for the action of an amenable semigroup of L_1 positive contractions using a criterion for *singular compactness*; that is, for the singularity of all the measures in a compact convex set of functionals in L_∞.

Consider the Lebesgue space $(X, m, \mathbb{B})$, and let G be a weakly continuous, locally compact semigroup of positive L_1 contractions. We denote the action of $g \in G$ on a function f by $g\,f$. Suppose that there exists a left invariant measure ν on G and increasing sequence $\{K_l\}_{l\in\mathbb{N}}$ of compact subsets satisfying the following (Følner) conditions:

(i) $0 < \nu(K_l) < \infty$,

(ii) for every $g \in G$,

$$\frac{\nu(K_l\,g \triangle K_l)}{\nu(K_l)} \to 0, \quad \frac{\nu(gK_l \triangle K_l)}{\nu(K_l)} \to 0 \quad \text{as } l \to \infty.$$

The space L_∞ ($L_\infty(X, m, \mathbb{B})$) is the Banach conjugate to L_1 and denote by L_∞^* the conjugate of L_∞. Let

$$A_l f = \frac{1}{\nu(K_l)} \int_{K_l} g\,f d\nu(g), \quad f \in L_1$$

and

$$A_l^* f = \frac{1}{\nu(K_l)} \int_{K_l} g^*\,f d\nu(g), \quad f \in L_\infty.$$

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A positive function $h \in L_\infty$ will be called **very weakly wandering** if

$$\|A_l^* h\|_\infty \to 0 \quad \text{as } l \to \infty.$$

Each functional $\mu \in L_\infty^*$ can be decomposed into $\mu = \mu_n + \mu_s$, where μ_n is absolutely continuous with respect to the measure m and μ_s is singular with respect to m. A functional μ is called a measure if it is positive and $\mu(\chi_X) = 1$, where χ_A denotes the characteristic function of the set $A \in \mathbb{B}$. Note that by the Banach-Alaoglu theorem, the set of all measures is compact in the weak* topology. At first we shall characterize the case when a convex subset of singular measures has only singular measures as limit points in the weak* topology. (In [OW] it is proved (theorem 6.1) that two disjoint compact convex sets of Borel measures can be separated by a G_δ set. We remark that the proof of Theorem 6.1 can be modified in order to give another proof of Theorem 1 and vice versa.)

THEOREM 1: *Let C be a convex set of singular measures. The following conditions are equivalent:*

(i) *The weak* closure of C consists of singular measures.*
(ii) *There exists a sequence of sets E_n, $E_n \in \mathbb{B}$, $E_n \uparrow X$ so that for every $\mu \in C$, $\mu(E_n) = 0$.*

In order to prove the theorem we need the following lemma. We denote by $\mathbb{R}^{+,n}$ the cone of positive vectors in $\mathbb{R}^n$.

LEMMA: *Let $C \subseteq \mathbb{R}^{+,n}$ be a convex compact subset,*

$$P_\delta = \{s \subset \{1,2,\dots,n\} \colon \frac{|s|}{n} \geq 1-\delta\}, \quad H_s(\delta) = \{\bar{x} \in \mathbb{R}^{+,n} \colon \sum_{i\in s} x_i < \delta\},$$

$$\bar{H}_s(\delta) = \mathbb{R}^{+,n} - H_s(\delta), \quad \bar{H}(\delta) = \bigcap_{s\in P_\delta} \bar{H}_s(\delta).$$

Assume that

$$C \subset \bigcup_{t\in P_\delta} H_t(\delta).$$

Then $C \subseteq H_t(2\delta)$ for some $t \in P_{2\delta}$.

Proof: It is easy to see that $\bar{H}(\delta)$ and $\bar{H}_s(\delta)$ are convex, so there exists a hyperplane $\sum \alpha_i x_i = 1$ such that for $\bar{x} \in C$ we have $\sum \alpha_i x_i \leq 1$ and for $\bar{y} \in \bar{H}(\delta)$ we have $\sum \alpha_i y_i > 1$. Letting $\bar{y} \in \bar{H}(\delta)$ tend to infinity we see that $\alpha_i \geq 0$. We claim that $Q = \{\bar{x} \in \mathbb{R}^{+,n} \colon \sum \alpha_i x_i \le 1\}$ is contained in $H_s(2\delta)$ for some $s \in P_{2\delta}$. Note that $Q \subset \bigcup_{t \in P_\delta} H_t(\delta)$. If Q is not in $H_s(2\delta)$ for any $s \in P_{2\delta}$, then the set $A = \{i \in \{1,2,\dots,n\} \colon \alpha_i < \frac{1}{2\delta}\}$ has $|A| > 2\delta n$; otherwise its complement $t = A^c$ belongs to $P_{2\delta}$ and $Q \subset H_t(2\delta)$. Consider the point $\bar{x}$ with coordinates

$$x_i = \begin{cases} \frac{1}{\alpha_i |A|} & \text{for } i \in A, \\ 0 & \text{for } i \notin A. \end{cases}$$

Then $\sum \alpha_i x_i = 1$ or $\bar{x} \in Q$. On the other hand, for every $t \in P_\delta$, $|t| > n - n\delta$ so that $|t \cap A| > \frac{1}{2|A|}$ and

$$\sum_{i \in t} x_i \ge \sum_{i \in t \cap A} \frac{1}{\alpha_i |A|} > \frac{2\delta |t \cap A|}{|A|} > \delta,$$

which contradicts the fact that $Q \subseteq \bigcup_{t \in P_\delta} H_t(\delta)$.

Proof of Theorem 1: (ii)$\Rightarrow$(i) is obvious.

(i) $\Rightarrow$(ii) Let K be a compact subset of singular measures. To every $F \in \mathbb{B}$ with $m(F) > 1 - \epsilon$ we associate $U_F = \{\mu : \mu(F) < \epsilon\}$. U_F is a cover of the compact set K. Let $U_{F_1}, \dots, U_{F_k}$ be a finite subcover. Let $G_1, \dots, G_n$ be a subpartition of $\bigcup_{j=1}^k F_j$ satisfying $m(G_i)/m(G_j) \le 1.1$ for each $1 \le i, j \le n$ and such that for each j there exists a subset $I_j \subset \{1, 2, \dots, n\}$ with

$$F_j = \bigcup_{i \in I_j} G_i.$$

Define a mapping $h \colon K \to \mathbb{R}^n$ as $h(\mu) = (\mu(G_1), \dots, \mu(G_n))$. Choosing δ appropriately one obtains a compact convex set $C = h(K)$ satisfying the conditions of the lemma. According to the lemma there exists a subset $t \subset \{1, \dots, n\}$ so that $|t| > n - 2n\epsilon$ and $\sum_{i \in t} \mu(G_i) < 2\epsilon$. So for $F = \cup_{i \in t} G_i$ we will have for every $\mu \in K, \mu(F) < \epsilon$ and $m(F) > 1 - 3\epsilon$. ∎

COROLLARY: *Let K be a bounded convex set of measures such that the restriction of K to some measurable subset $E_0 \in \mathbb{B}$ consists of singular measures and for every $E \in \mathbb{B}$, $E \subseteq E_0$ with $m(E) \neq 0$ there exists some $\mu \in K$ such that $\mu(E) > 0$. Then the closure of K in the weak* topology contains a measure μ_0 whose normal component $\mu_{0,n}$ satisfies $\mu_{0,n}(E') > 0$ for some $E' \subset E$, $E' \in \mathbb{B}$, with $m(E') > 0$.*

The following theorem gives the Neveu decomposition for the action of an amenable semigroup. We remark that the proof is new and considerably shorter even in the single operator case [Kr, th. 6.3.9 and th. 3.4.6].

THEOREM 2: *Let G be as before. The following conditions on a set $E \in \mathbb{B}$ are equivalent:*

(i) *There exists a G-invariant function $f \in L_1$ with support E, and the support of every G-invariant function is contained in E.*

(ii) *There exists a very weakly wandering function $h \in L_\infty$ with support $X \setminus E$, and the support of each very weakly wandering function is contained in $X \setminus E$.*

Proof: 1. Let $\bar{x}$ be a weak* limit point of the sequence $A_l^* x$. Then $A_l^* x$ converges strongly and for every $g \in G$, $g^* \bar{x} = \bar{x}$. Indeed,

$$\|A_l^* g^* x - A_l^* x\| \le \|x\| \nu(gK_l \Delta K_l)/\nu(K_l) \to 0.$$

For every $\epsilon > 0$ there exist $\lambda_1, \dots, \lambda_j > 0$ with $\sum \lambda_i = 1$ and $\|\sum \lambda_i A_{l_i}^* x - \bar{x}\| < \epsilon$. Each $A_{l_i}^* x$ is in the norm closure of the convex hull of the $g^* x$ and so $\|A_l^* \, A_{l_i}^* \, x - A_l^* \, x\| \to 0$ as $l \to \infty$. This gives

$$\|A_l^* x - \bar{x}\| \le \|A_l^* (\sum \lambda_i A_{l_i}^* x - \bar{x})\| + \|A_l^* \sum (\lambda_i A_{l_i}^* x - x)\| \to 0 \quad \text{as } l \to \infty.$$

2. Let $\mu \in L_\infty^{*,+}, \mu(X) = 1$. If $g^{**}\mu = \mu$, then for μ_n and μ_s we have $g^{**}\mu_n = \mu_n$, $g^{**}\mu_s = \mu_s$. Indeed, $g^{**}\mu_n \le \mu$ implies $g^{**}\mu_n \le \mu_n$. Since $\|g^{**}\mu_s\| \le \|\mu_s\|$ and $\|g^{**}\mu_s\| + \|g^{**}\mu_n\| = \|g^{**}\mu\| = \|\mu_s\| + \|\mu_n\|$, and, moreover, $\|g^{**}\mu_s\| \le \|\mu_s\|$ we must have $\|g^{**}\mu_s\| = \|\mu_s\|$. This implies $g^{**}\mu_s = \mu_s$ and hence $g^{**}\mu_n = \mu_n$.

3. There exists a G-invariant absolutely continuous measure μ whose support is maximal among such measures. This follows from the σ-finiteness of X. The support of μ is the set E in (i).

4. Let $P \subset X \setminus E$. We claim that there exists $P' \subseteq P$, $P' \in \mathbb{B}$, $m(P') > 0$ such that every invariant measure vanishes in P'. Otherwise there exists $P_0 \in \mathbb{B}$ such that for every $P' \subset P_0$, $P' \in \mathbb{B}$, $m(P') > 0$ there exists an invariant measure μ with $\mu(P') > 0$. By the definition of E the restriction of μ to $X \setminus E$ is singular. The corollary to Theorem 1 now implies the existence of an invariant measure μ_1 with $\mu_{1,n}(P'') > 0$ for some $P'' \subseteq P$ satisfying $\mu(P'') > 0$. By (3) $\mu_{1,n}$ is also invariant; but since it is absolutely continuous, this contradicts the definition of E.

5. Thus there exists a sequence $E_n \uparrow X \setminus E$ with $\mu(E_n) = 0$ for every invariant measure μ. Let $\nu \in L_\infty^*$ and $A_\alpha^{**}\nu \to \nu_0$ be a convergent subnet in weak* topology. Since $\nu_0(E_n) = 0$ we have

$$A_\alpha^{**}\nu(E_n) \to 0$$

for every subnet α for which $A_\alpha^{**}\nu$ does converge . So $A_l^*(\chi_{E_n}) \to 0$ weakly* or $\|A_l^*(\chi_{E_n})\|_\infty \to 0$ or χ_{E_n} is a very weakly wandering function. If we set $h = \sum 2^{-n}\chi_{E_n}$ then h satisfies the condition of (ii).

(ii)$\Rightarrow$(i) is obvious. ∎

Theorem 2 implies the convergence in measure of averages $A_l f$ (compare [Kr], Theorem 6.3.10).

COROLLARY: *Let A_l be as before. For $f \in L_1$ the sequence $A_l f$ converges in measure.*

Sketch of proof: On $X \setminus E$ the convergence in measure follows from existence of very weakly wandering function with support $X \setminus E$. On E and for f from L_2 the result follows from the Mean Ergodic Theorem [Gr]. By an approximation argument, one can extend this to L_1.

COROLLARY: *If the set of G-invariant measures contains only one absolutely continuous measure then there exists $E \in \mathbb{B}$, $m(E) = 0$ such that every singular measure is equal to 0 on $X \setminus E$.*

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